Dynamo effects in magnetized ideal-plasma cosmologies

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The excitation of cosmological perturbations in an anisotropic cosmological model and in the presence of a homogeneous magnetic field has been studied, using the ideal magnetohydrodynamic (MHD) equations. In this case, the system of partial differential equations which governs the evolution of the magnetized cosmological perturbations can be solved analytically. Our results verify that fast-magnetosonic modes propagating normal to the magnetic field, are excited. But, what's most important, is that, at late times, the *magnetic-induction contrast* ($\delta B/B$) grows, resulting in the enhancement of the ambient magnetic field. This process can be particularly favored by condensations, formed within the plasma fluid due to gravitational instabilities.

I. INTRODUCTION

Magnetic fields are known to have a widespread presence in our Universe, being a common property of the intergalactic medium in galaxy clusters [1], while, reports on Faraday rotation imply significant magnetic fields in condensations at high redshifts [2]. Large-scale magnetic fields and their potential implications for the formation and the evolution of the observed structures, have been the subject of continuous theoretical investigation in the past [3] - [6]. It became clear that if magnetism has a cosmological origin, it could have affected the evolution of the Universe [7]. Today, there are several scenarios for the generation of primordial magnetic fields [8]. The majority of the recent studies use a Friedmann - Robertson - Walker (FRW) model to represent the evolving Universe and super-impose a large-scale magnetic field [9]. In other words, the magnetic field is assumed to be too weak to destroy the FRW isotropy and any potential anisotropy induced by it, is treated as a perturbation [10].

However, mathematically speaking, the spatial isotropy of the FRW Universe is not compatible with the presence of large-scale magnetic fields. In fact, an anisotropic cosmological model can and should be imposed for the treatment of magnetic fields whose *coherent length* is comparable to the horizon length [11]. Therefore, although current observations give a strong motivation for the adoption of a FRW model, the effects one may lose by neglecting the large-scale anisotropy induced by the background magnetic field, should be investigated. Not to mention that the anisotropy of the unperturbed model facilitates a closer study of the coupling between magnetism and geometry.

In a recent paper [12], which hereafter is referred to as Paper I, the evolution of a magnetized, resistive plasma in an anisotropic space-time has been studied numerically. The corresponding results suggested that fast-magnetosonic waves grow steeply with time and saturated at high values, due to the resistivity. Nevertheless, numerics indicated also that growing modes are present

even in the limit of zero resistivity (the *ideal plasma* case). Therefore, in the present article we consider the same model, but, this time in the limit of the ideal MHD approximation; namely, the assumption that the magnetic field is frozen into an effectively infinitely conductive cosmic medium (a fluid of zero resistivity). As we find out, in this case, the evolution of the cosmological perturbations can be treated analytically.

Following the procedure described in Paper I, we begin with a resistive plasma, driving the dynamics of an anisotropic space-time, in the presence of a time-dependent magnetic field. This dynamical system is subsequently perturbed by small-scale fluctuations and we study their interaction with the curved background, searching for imprints on the temporal evolution of the perturbations' amplitude. In particular:

In Section II, we present the system of the field equations describing the model under consideration and the corresponding zeroth-order solution. Accordingly, in Section III, we extract the first-order perturbed equations and, confining ourselves in the limit of zero-resistivity, we derive an analytic solution for the magnetized cosmological perturbations. Our results verify that, fast-magnetosonic modes are excited within the ideal plasma. At late times, the magnetic induction contrast ($\delta B/B$) grows, resulting in the enhancement of the ambient magnetic field (dynamo effect). This effect is particularly favored by condensations that can be formed within the anisotropic fluid, due to gravitational instabilities.

II. A MAGNETIZED ANISOTROPIC COSMOLOGY

We consider an axially-symmetric Bianchi-Type I cosmological model, driven by an anisotropic and resistive perfect fluid, in the presence of a time-dependent magnetic field, $\vec{B} = B(t)\hat{x}$. In the system of units where $\hbar = 1 = c$, the corresponding line-element is written in

the form

$$ds^2 = -dt^2 + R^2(t)dx^2 + S^2(t)[dy^2 + dz^2] \qquad (1)$$

where, the dimensionless scale factors $R(t)$ and $S(t)$ are functions of the time coordinate.

The evolution of a curved space-time in the presence of matter and an e/m field, is determined by the gravitational field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G \mathcal{T}_{\mu\nu} \qquad (2)$$

together with the energy-momentum conservation law

$$\mathcal{T}^{\mu\nu}_{;\nu} = 0 \qquad (3)$$

and Maxwell's equations

$$F^{\mu\nu}_{;\nu} = 4\pi J^\mu, \qquad (4)$$
$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \qquad (5)$$

In Eqs (2) - (5), Greek indices refer to the four-dimensional space-time (in accordance, Latin indices refer to the three-dimensional spatial section) and the semicolon denotes covariant derivative. Furthermore, $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the Ricci tensor and the scalar curvature with respect to the background metric $g_{\mu\nu}$, while G is Newton's gravitational constant. Eventually, $F^{\mu\nu}$ is the antisymmetric tensor of the e/m field and J^μ is the corresponding current-density.

The energy-momentum tensor involved, consists of two parts; namely,

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu\nu}_{fluid} + \mathcal{T}^{\mu\nu}_{em} \qquad (6)$$

The first part, is due to an anisotropic perfect-fluid source of the form

$$\mathcal{T}^{\mu\nu}_{fluid} = \rho u^0 u^0 + p_i u^i u^i + p_i g^{ii} \qquad (7)$$

where, $\rho(t)$ is the energy-density, $p_i(t)$ are the components of the anisotropic pressure and the axial symmetry of the metric (1) implies that $p_2(t) = p_3(t)$. In Eq (7), $u^\mu = dx^\mu/ds$ is the fluid's four-velocity, satisfying the conditions $u_\mu u^\mu = -1$ and $h^{\mu\nu} u_\mu = 0$, with $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ being the projection tensor.

The second part of $\mathcal{T}^{\mu\nu}$, is due to the ambient e/m field

$$\mathcal{T}^{\mu\nu}_{em} = \frac{1}{4\pi}(F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \qquad (8)$$

In the absence of electric fields (something that is strongly suggested by Paper I), the only non-zero components of the Faraday tensor in the curved space-time (1) are

$$F^{23} = \frac{B^x}{S^2} = -F^{32} \qquad (9)$$

As regards the current-density J^μ, it can be determined by the invariant form of Ohm's law

$$J^\mu = n_e e u^\mu + \frac{1}{\eta}F^{\mu\nu}u_\nu \qquad (10)$$

where, n_e is the locally measured charge-density and η is the electric resistivity, in units of *time*. It is reasonable to assume that, locally, the fluid has zero net-charge. In this case, Eq (10) reduces to $J^\mu = \frac{1}{\eta}F^{\mu\nu}u_\nu$ and the identity $J^\mu_{;\mu} = 0$ holds, as a consequence of the Maxwell equations. We have to point out that, in the zero-resistivity limit Eq (4) reduces, through Eq (10), to

$$F^{\mu\nu}u_\nu = 0 \qquad (11)$$

[clf Eq (B2) of Paper I]. In other words, for $\eta = 0$, we are left only with the *convective field* [13].

Following the procedure described in detail in Paper I, we look for an axially-symmetric Bianchi-Type I cosmological solution to the Einstein-Maxwell equations (2) - (5), representing the zeroth-order solution of our problem. In this case, Eqs (2) reduce to

$$2\left(\frac{\dot{R}\dot{S}}{RS}\right) + \left(\frac{\dot{S}}{S}\right)^2 = 8\pi G\rho(t) + GB^2(t)$$

$$-2\frac{\ddot{S}}{S} - \left(\frac{\dot{S}}{S}\right)^2 = 8\pi G p_1(t) - GB^2(t) \qquad (12)$$

$$-\left(\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S}\right) - \left(\frac{\dot{R}\dot{S}}{RS}\right) = 8\pi G p_2(t) + GB^2(t)$$

where, the dot denotes time-derivative. For $\vec{E} = \vec{0}$ and $\vec{B}(t) \;//\; \hat{x}$, Eq (4) vanishes identically and Eq (5) yields

$$\partial_t[S^2 B(t)] = 0 \;\Rightarrow\; B(t) = \frac{B_0}{S^2} \qquad (13)$$

where, B_0 is the initial value of the magnetic induction. Eq (13) has a clear physical interpretation: The magnetic flux through a comoving surface normal to the direction of the magnetic field, *is conserved*. Furthermore, the continuity equation (3) results in

$$\partial_t[\rho(t) + \frac{B^2}{8\pi}] + \frac{\dot{R}}{R}[p_1(t) - \frac{B^2}{8\pi}] + 2\frac{\dot{S}}{S}[p_2(t) + \frac{B^2}{8\pi}]$$

$$+(\frac{\dot{R}}{R} + 2\frac{\dot{S}}{S})[\rho(t) + \frac{B^2}{8\pi}] = 0 \qquad (14)$$

and the particles' number conservation law, reads

$$\dot{\rho} + (\frac{\dot{R}}{R} + 2\frac{\dot{S}}{S})\rho = 0 \;\Rightarrow\; \rho(t) = \frac{\rho_0}{RS^2} \qquad (15)$$

where, ρ_0 is the initial energy-density. In Paper I, we showed that the system of Eqs (12) - (15) admits the

exact solution

$$R(t) = (\frac{t}{t_0}), \quad S(t) = (\frac{t}{t_0})^{\frac{1}{2}}$$

$$\rho(t) = \rho_0(\frac{t_0}{t})^2 \quad B(t) = B_0(\frac{t_0}{t}) \quad (16)$$

$$p_1(t) = p_{10}(\frac{t_0}{t})^2, \quad p_2(t) = -p_{20}(\frac{t_0}{t})^2$$

where, the index "0" stands for the corresponding values at $t = t_0$, which marks the beginning of the interaction between magnetized plasma and curved space-time. Solution (16), describes the evolution of an anisotropic cosmological model, in which, the large-scale anisotropy along the $\hat{x}$-axis, is due to the presence of an ambient magnetic field. The combination of Eqs (12) and (16) indicates that, initially, the equation of state for the matter-energy content reads

$$p_0 = \frac{1}{5}(\rho_0 + 6\frac{B_0^2}{8\pi}) \quad (17)$$

and, therefore, as regards the perfect fluid itself, we obtain $p_0 = \frac{1}{5}\rho_0$. This *soft equation of state* enlists the curved space-time (1) among the *semi-realistic* cosmological models of Bianchi Type I. These models are crude, first-order approximations to the actual Universe, when we use currently available theories and observations [14].

III. COSMOLOGICAL PERTURBATIONS IN AN IDEAL PLASMA FLUID

For every dynamical system, much can be learnt by investigating the possible modes of small-amplitude oscillations or waves. A plasma is physically much more complicated than an ideal gas, especially when there is an externally applied magnetic field. As a result, a variety of small-scale perturbations may appear.

Following the formalism of Paper I, we introduce first-order perturbations to the Einstein-Maxwell equations, by decomposing the physical variables of the fluid as

$$\rho(t,z) = \rho(t) + \delta\rho(t,z) \quad (18)$$
$$p_x(t,z) = p_1(t)$$
$$p_y(t,z) = p_2(t) - \delta p(t,z) \quad (19)$$
$$p_z(t,z) = p_2(t) + \delta p(t,z)$$

and we insert the perturbed values (18) and (19) into Eqs (12) - (15), neglecting all terms higher or equal than the second order. The pressure perturbation $\delta p(t,z)$ introduces a longitudinal acoustic mode, propagating along the $\hat{z}$-direction

$$\delta p(t,z) = C_s^2 \, \delta\rho(t,z) \quad (20)$$

where, $C_s = \frac{1}{\sqrt{5}}$ is the speed of sound. The four-velocity of the plasma fluid is perturbed around its comoving value, $u^\mu = (1,0,0,0)$, as

$$u^\mu(t,z) = (1 + \delta u^0(t,z), 0, 0, \delta u^z(t,z)) \quad (21)$$

Then, the condition $u_\mu u^\mu = -1$, to the first leading order, implies

$$\delta u^0(t,z) = 0 \quad (22)$$

and, therefore, $u^3(t,z) = \delta u^z(t,z)$. Accordingly, $\rho(t,z)u^3(t,z) = \rho(t)\delta u^z(t,z) + O_2$.

As regards the perturbations of the e/m field, we consider that they correspond to a transverse e/m wave, propagating along the $\hat{z}$-axis ($\vec{k}//\hat{z}$); namely,

$$\vec{E}(t,z) = \delta E^y(t,z)\hat{y} \quad (23)$$
$$\vec{B}(t,z) = B(t)\hat{x} + \delta B^x(t,z)\hat{x} \quad (24)$$

Therefore, now, the non-zero components of the Faraday tensor in the curved space-time (1) are modified as follows

$$F^{02} = \frac{1}{S}\,\delta E^y(t,z) = -F^{20}$$
$$F^{23} = \frac{1}{S^2}\,[B(t) + \delta B^x(t,z)] = -F^{32} \quad (25)$$

In the search for dynamo effects within the linear regime, we should stress that magnetic fields (as well as their inhomogeneities) are not created by first-order terms in the metric perturbations: Their production involves electric currents generated by the rotational component of the velocity of the plasma constituents, along the lines of the so-called *vorticity effect* [15], [16] and this component arises only at the second-order approximation [17] - [22].

Within the limits of linear analysis, excitation of cosmological perturbations in a homogeneous and anisotropic cosmological model is basically a *kinematic effect*, in the sense that the self-gravitation of the fluctuations is unimportant (e.g. see [23], pp. 501 - 506 and references therein). In this case, perturbations' growth may arise mostly due to their motion in the anisotropic background.

Therefore, as far as the enhancement of MHD perturbations in the anisotropic space-time (1) is concerned, we may neglect the first order corrections of the metric, admitting the so-called *Cowling approximation* [24]. In other words, in what follows we treat the MHD perturbations as very low-frequency (small-amplitude) waves propagating in an anisotropically expanding medium, without interacting with the curved space-time unless the linear regime breaks down. Accordingly, the evolution of the perturbed quantities is governed only by the energy-momentum tensor conservation, together with Maxwell's equations.

The linearly-independent, first-order perturbed Einstein-Maxwell equations in the curved background

(1), first extracted in Paper I, read

$$\partial_t(\delta\rho) + \rho(t)\partial_z(\delta u^z) + \delta\rho\left(\frac{\dot{R}}{R} + 2\frac{\dot{S}}{S}\right) = 0 \quad (26)$$

$$-\partial_t(\delta E^y) + \frac{1}{S}\partial_z(\delta B^x) - \delta E^y\left(\frac{\dot{R}}{R} + \frac{\dot{S}}{S}\right) =$$

$$= 4\pi\frac{1}{\eta}[\delta E^y + SB(t)\delta u^z] \quad (27)$$

$$\partial_t(S^2\delta B^x) - S\partial_z(\delta E^y) = 0 \quad (28)$$

$$\partial_t[\rho(t)\delta u^z - \frac{1}{4\pi S}B(t)\delta E^y] + \frac{1}{S^2}\partial_z[\delta p + \frac{1}{4\pi}B(t)\delta B^x] +$$

$$+ \left(\frac{\dot{R}}{R} + 2\frac{\dot{S}}{S}\right)[\rho(t)\delta u^z - \frac{1}{4\pi S}B(t)\delta E^y] = 0 \quad (29)$$

Confining ourselves in an ideal plasma, Eq (27) results in

$$\delta E^y = -SB(t)\delta u^z \quad (30)$$

verifying that, for $\eta = 0$, one is left only with the perturbations of the *convective field*. Differentiating Eq (30) with respect to z

$$\partial_z(\delta E^y) = -SB(t)\partial_z(\delta u^z) \quad (31)$$

and taking into account the conservation of the magnetic flux, Eq (28) is written in the form

$$\partial_t(S^2\delta B^x) = -B_0\partial_z(\delta u^z) \quad (32)$$

On the other hand, with the aid of the particles' number conservation law, Eq (26) reads

$$\partial_z(\delta u^z) = -\frac{1}{\rho_0}\partial_t[RS^2(\delta\rho)] \quad (33)$$

and therefore, Eq (32) yields

$$\partial_t(S^2\delta B^x) = \frac{B_0}{\rho_0}\partial_t[RS^2(\delta\rho)] \quad (34)$$

We integrate Eq (34) with respect to t, to obtain

$$\delta B^x = \frac{B_0}{\rho_0}R(t)\,\delta\rho + \frac{1}{S^2}f(z) \quad (35)$$

Without loss of generality, we may impose that $f(z) = 0$. In fact, in an expanding Universe, the term $f(z)/S^2$ eventually becomes negligible due to the cosmological redshift. Therefore, as regards the temporal evolution of δB^x, to set $f(z) = 0$ simply corresponds to a translation in the origin of time, to the point where $f(z)/S^2$ vanishes. Accordingly,

$$\delta B^x = \frac{B_0}{\rho_0}R(t)\,\delta\rho \quad (36)$$

With the aid of Eqs (13) and (15), Eq (36) results in

$$\frac{\delta B}{B(t)} = \frac{\delta\rho}{\rho(t)} \quad (37)$$

We observe that, in an anisotropic cosmological model which is driven by ideal plasma, the *magnetic induction contrast* is equal to the *energy-density contrast*. In other words, the magnetic field perturbation amplifies in tune with the energy-density perturbation and therefore, any Jeans instability (condensations that can be formed inside the plasma fluid due to an unstable growth in $\delta\rho$) results in the increase of δB^x and, hence, acts in favor of *dynamo effects*. In fact, Eq (37) was first obtained in [11], regarding a Bianchi Type I model filled with ideal plasma, within the context of the so-called *covariant formalism* (e.g. see Eq (98) of [11]). Coincidence of these results, although in [11] perturbations of the metric were taken into account, justifies the assumption that the evolution of the cosmological perturbations in an anisotropic background is a purely kinematic effect [23].

Now, the system of the first-order perturbation equations (26) - (28) is written in the form

$$\delta B^x = \frac{B_0}{\rho_0}R(t)\,\delta\rho$$

$$\partial_z(\delta u^z) = -\frac{1}{\rho_0}\partial_t[RS^2(\delta\rho)] \quad (38)$$

$$\partial_z(\delta E^y) = \frac{1}{S}\left(\frac{B_0}{\rho_0}\right)\partial_t[RS^2(\delta\rho)]$$

where, the last equation has resulted from the combination of Eqs (31) and (33), with the aid of the magnetic flux conservation. From Eqs (38) it becomes evident that the behavior of the velocity and the electric field perturbations is modulated mostly due to the inherent anisotropy of the expanding medium. We also note that the evolution of every perturbation quantity on the lhs, involves the evolution of $\delta\rho$. The first-order perturbation of the energy-density is accordingly determined by Eq (29), which, upon consideration of Eqs (30) and (36), reduces to

$$\rho_0\partial_t\left[(1 + u_A^2\frac{R}{S^2})\delta u^z\right] + R(C_s^2 + u_A^2\frac{R}{S^2})\partial_z(\delta\rho) = 0 \quad (39)$$

where, $u_A^2 = B_0^2/4\pi\rho_0$ is the (dimensionless) Alfvén velocity. However, as regards the cosmological model under study, we have $R(t) = S^2(t)$ and, therefore, Eq (39) takes on its final form

$$\rho_0(1 + u_A^2)\partial_t(\delta u^z) + R(C_s^2 + u_A^2)\partial_z(\delta\rho) = 0 \quad (40)$$

We differentiate Eq (33) with respect to t, to obtain

$$\partial_t\partial_z(\delta u^z) = -\frac{1}{\rho_0}\partial_t^2[RS^2(\delta\rho)] \quad (41)$$

Accordingly, we differentiate Eq (40) with respect to z, to obtain

$$\rho_0(1 + u_A^2)\partial_z\partial_t(\delta u^z) + R(t)(C_s^2 + u_A^2)\partial_z^2(\delta\rho) = 0 \quad (42)$$

The combination of Eqs (41) and (42) results in

$$-(1 + u_A^2)\partial_t^2 \Psi(t,z) + \frac{1}{S^2}(C_s^2 + u_A^2)\partial_z^2 \Psi(t,z) = 0 \quad (43)$$

where, we have set

$$\Psi(t,z) = RS^2(\delta\rho) \quad (44)$$

To solve Eq (43), we use the method of separation of variables, considering

$$\Psi(t,z) = T(t)Z(z) \quad (45)$$

Now, the equation which governs the evolution of $\Psi(t,z)$, is written in the form

$$\frac{1 + u_A^2}{C_s^2 + u_A^2}\left(\frac{t}{t_0}\right)\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t) = \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) \quad (46)$$

implying that both parts are equal to an arbitrary constant (λ); namely,

$$\frac{\partial^2}{\partial z^2}Z(z) = \lambda Z(z) \quad (47)$$

and

$$\frac{\partial^2}{\partial t^2}T(t) = \lambda\left(\frac{t_0}{t}\right)\frac{C_s^2 + u_A^2}{1 + u_A^2}\,T(t) \quad (48)$$

In Paper I we have showed that the only perturbation modes admitted in an ideal plasma are the magnetosonic waves

$$(1 + u_A^2)\omega_0^2 = (C_s^2 + u_A^2)k^2 \quad (49)$$

(clf Eq (48) of [12]) where, k is the comoving wavenumber and ω_0 is the corresponding angular-frequency of the wave. In order to incorporate these modes in our analysis, we need to impose

$$\lambda = -k^2 \quad (50)$$

in which case, Eqs (47) and (48) result in

$$Z(z) \sim e^{ikz} \quad (51)$$

and

$$\frac{\partial^2}{\partial t^2}T(t) + \omega_0^2\left(\frac{t_0}{t}\right)T(t) = 0 \quad (52)$$

Eq (52) admits formal Bessel-type solutions [25]

$$T(x) = \sqrt{x}\,[c_1 J_1(2\omega_0 t_0\sqrt{x}) + c_2 Y_1(2\omega_0 t_0\sqrt{x})] \quad (53)$$

where, c_1 and c_2 are arbitrary integration constants and we have set

$$1 \leq x = \frac{t}{t_0} < \infty \quad (54)$$

Therefore, the energy-density perturbation reads

$$\delta\rho = \frac{1}{x^{3/2}}[c_1 J_1(2\omega_0 t_0\sqrt{x}) + c_2 Y_1(2\omega_0 t_0\sqrt{x})]e^{ikz} \quad (55)$$

To find the functional form of the other perturbation quantities, we use Eqs (38) and take into account the following recurrence relations of the Bessel functions [25]

$$J_\nu'(\phi) = J_{\nu-1}(\phi) - \frac{\nu}{\phi}J_\nu(\phi)$$

$$Y_\nu'(\phi) = Y_{\nu-1}(\phi) - \frac{\nu}{\phi}Y_\nu(\phi) \quad (56)$$

where, a prime denotes differentiation with respect to the argument ϕ. Accordingly, we obtain

$$\delta u^z = \frac{\omega_0}{k\rho_0}[c_1 J_0(2\omega_0 t_0\sqrt{x}) + c_2 Y_0(2\omega_0 t_0\sqrt{x})]e^{i(kz+\frac{\pi}{2})} \quad (57)$$

$$\delta E^y = \frac{\omega_0 B_0}{k\rho_0\sqrt{x}}[c_1 J_0(2\omega_0 t_0\sqrt{x}) + c_2 Y_0(2\omega_0 t_0\sqrt{x})]e^{i(kz-\frac{\pi}{2})} \quad (58)$$

$$\delta B^x = \frac{B_0}{\rho_0\sqrt{x}}[c_1 J_1(2\omega_0 t_0\sqrt{x}) + c_2 Y_1(2\omega_0 t_0\sqrt{x})]e^{ikz} \quad (59)$$

With the particular choice $c_2 = -ic_1$, the time-dependent amplitude of the cosmological perturbations can be written in the form of Hankel functions; namely,

$$\delta\rho = \frac{1}{x^{3/2}}\,c_1\,H_1^{(2)}(2\omega_0 t_0\sqrt{x})\,e^{ikz} \quad (60)$$

$$\delta u^z = \frac{\omega_0}{k\rho_0}\,c_1\,H_0^{(2)}(2\omega_0 t_0\sqrt{x})\,e^{i(kz+\frac{\pi}{2})} \quad (61)$$

$$\delta E^y = \frac{\omega_0 B_0}{k\rho_0}\,\frac{1}{x^{1/2}}\,c_1\,H_0^{(2)}(2\omega_0 t_0\sqrt{x})\,e^{i(kz-\frac{\pi}{2})} \quad (62)$$

$$\delta B^x = \frac{B_0}{\rho_0}\,\frac{1}{x^{1/2}}\,c_1\,H_1^{(2)}(2\omega_0 t_0\sqrt{x})\,e^{ikz} \quad (63)$$

Eqs (60) - (63) possess a very interesting asymptotic behavior at both large and small arguments (e.g. see [26]).

Although it is quite clear what do we mean by the large-argument behavior; it corresponds to what the perturbations' may look like at late times $(t \to \infty)$, the corresponding small-argument limit is rather ambiguous, since $t \geq t_0$. At this point we should emphasize that the magnetosonic waves are, in fact, *low frequency modes*. Therefore, we expect that ω_0 should be much smaller than the characteristic frequency for the matter and energy to shift about, along the direction of propagation, at $t = t_0$; namely, the Hubble parameter across the z-axis. Accordingly,

$$\omega_0 \ll H_S(t_0) = \frac{1}{2t_0} \quad \Rightarrow \quad 2\omega_0 t_0 \ll 1 \quad (64)$$

Therefore, the small-argument behavior simply corresponds to small values of t after t_0.

A. Asymptotic behavior for small values of the argument

In this case, which corresponds to an early-time approximation ($x \simeq 1$ and $2\omega_0 t_0\sqrt{x} \ll 1$), the cosmological

perturbations read

$$\delta\rho = \frac{c_1}{\pi\omega_0 t_0}\,\frac{1}{x^2}\,e^{i(kz+\frac{\pi}{2})} \tag{65}$$

$$\delta u^z = \frac{2\omega_0}{\pi k\rho_0}\,c_1\,\ln\frac{1}{\omega_0 t_0\sqrt{x}}\,e^{i(kz+\pi)} \tag{66}$$

$$\delta E^y = \frac{2\omega_0 B_0}{\pi k\rho_0}\,c_1\,\frac{1}{\sqrt{x}}\,\ln\frac{1}{\omega_0 t_0\sqrt{x}}\,e^{ikz} \tag{67}$$

$$\delta B^x = \frac{B_0}{\rho_0}\,\frac{c_1}{\pi\omega_0 t_0}\,\frac{1}{x}\,e^{i(kz+\frac{\pi}{2})} \tag{68}$$

We observe that, both the energy-density and the magnetic field perturbation decrease adiabatically, due to the cosmological expansion [clf Eq (16)]. The amplitude of the magnetic induction contrast is written in the form

$$|\frac{\delta B}{B}|_{init} = \frac{c_1}{\pi\omega_0 t_0}\,\frac{1}{\rho_0} = |\frac{\delta\rho}{\rho}|_{init} \tag{69}$$

i.e. it is equal to the constant value of the energy-density contrast [a not unexpected result, clf Eq (37)].

B. Asymptotic behavior for large values of the argument

In this case, which corresponds to a late-time approximation ($x \to \infty$ and $2\omega_0 t_0\sqrt{x} \gg 1$), the cosmological perturbations read

$$\delta\rho = \frac{1}{x^{7/4}}\,\frac{c_1}{\sqrt{\pi\omega_0 t_0}}\,e^{i(kz-2\omega_0 t_0\sqrt{x}+\frac{3\pi}{4})} \tag{70}$$

$$\delta u^z = \frac{1}{x^{1/4}}\,\frac{\omega_0}{k\rho_0}\,\frac{c_1}{\sqrt{\pi\omega_0 t_0}}\,e^{i(kz-2\omega_0 t_0\sqrt{x}+\frac{3\pi}{4})} \tag{71}$$

$$\delta E^y = \frac{1}{x^{3/4}}\,\frac{\omega_0 B_0}{k\rho_0}\,\frac{c_1}{\sqrt{\pi\omega_0 t_0}}\,e^{i(kz-2\omega_0 t_0\sqrt{x}-\frac{\pi}{4})} \tag{72}$$

$$\delta B^x = \frac{1}{x^{3/4}}\,\frac{B_0}{\rho_0}\,\frac{c_1}{\sqrt{\pi\omega_0 t_0}}\,e^{i(kz-2\omega_0 t_0\sqrt{x}+\frac{3\pi}{4})} \tag{73}$$

Now, the amplitude of the magnetic induction contrast is written in the form

$$|\frac{\delta B}{B}|_{final} = \frac{c_1/\rho_0}{\sqrt{\pi\omega_0 t_0}}\,(\frac{t}{t_0})^{1/4} \tag{74}$$

and the same relation holds for the energy-density contrast, as well. It is worth noting that, while initially $(\delta B/B)$ acquires a constant value, at late times, it results in an increasing function of time. In fact, combining Eqs (69) and (74), we obtain

$$|\frac{\delta B}{B}|_{final} = \sqrt{\pi\omega_0 t_0}\,(\frac{t}{t_0})^{1/4}\,|\frac{\delta B}{B}|_{init} \tag{75}$$

Since both $(\delta B/B)$ and $(\delta\rho/\rho)$ are increasing functions of time, there is a characteristic time, t_c, at which

$$\frac{\delta B}{B}\,,\,\frac{\delta\rho}{\rho} \simeq 1 \tag{76}$$

and the linear analysis breaks down. However, the growth of δB with respect to $B(t)$ is, in fact, a very slow process. We may calculate explicitly the temporal limits of the linear approach, in a realistic setting.

Small-angle anisotropy in the CMRB implies that, for adiabatic perturbations along the recombination epoch (t_R), one has [27]

$$(\frac{\delta T}{T})_R = \frac{1}{3}\,(\frac{\delta\rho}{\rho})_R \tag{77}$$

Accordingly, at $t = t_R$ we have

$$|\frac{\delta B}{B}|_{final} = |\frac{\delta\rho}{\rho}|_{final} \lesssim 3\,(\frac{\delta T}{T})_R \tag{78}$$

where, the second step follows because the MHD modes can constitute no more than 100% of the observed CMRB anisotropy. With the aid of Eq (78) we may determine the arbitrary constant involved in Eq (74); namely

$$c_1 = 3\,(\frac{\delta T}{T})_R\,\rho_0\,\sqrt{\pi\omega_0 t_0}\,(\frac{t_0}{t_R})^{1/4} \tag{79}$$

and hence, in terms of t_R, Eq (74) reads

$$|\frac{\delta B}{B}|_{final} \lesssim 3\,(\frac{\delta T}{T})_R\,(\frac{t}{t_R})^{1/4} \tag{80}$$

We have to point out that, in Eq (80) the temporal limits are $t_0 \le t \le t_R$, since this mechanism does not apply after t_R. In other words, no large-scale inhomogeneities of the magnetic field can be formed after recombination, when electrons and protons combine to form neutral hydrogen and radiation decouples from matter (in connection, see also [21]).

Nevertheless, extrapolating this result beyond t_R, we may take a good idea of how slow the enhancement of these magnetic fluctuations may be. In this case, according to Eq (80), the characteristic time at which $\delta B \simeq B$ and the linear approach is no longer valid, reads

$$t_c \gtrsim \frac{1}{81(\frac{\delta T}{T})_R^4}\,t_R \tag{81}$$

Current observations (e.g. see [28]) imply that

$$(\frac{\delta T}{T})_R \sim 10^{-5} \tag{82}$$

and therefore

$$t_c \gtrsim 10^{18}\,t_R \tag{83}$$

Notice that, even if Eq (74) [or (80)] was valid also in the matter-dominated era, the amplification of δB should have been continued up to 10^{18} times the recombination epoch (10^{14} times the age of the Universe) before it can become comparable to B_0.

Therefore, dynamo effects (enhancement of the ambient magnetic field) do take place in an anisotropic background, with the linear approach being valid at high accuracy. According to Eq (37), this is most certainly true in the presence of large scale condensations within the cosmic fluid.

IV. CONCLUSIONS

In the present article, we explore the possibility of enhancing a primordial (*seed*) magnetic field, by taking advantage of the large-scale anisotropy created by it, within a fluid of infinite conductivity. To do so, we study the evolution of (low-frequency) magnetosonic waves in an anisotropically expanding ideal plasma and in the presence of a homogeneous magnetic field along the $\hat{x}$-direction.

Following the procedure described in Paper I [12], we begin with a resistive fluid driving the dynamics of the curved space-time (the so-called zeroth-order solution). This dynamical system is subsequently perturbed by small-scale fluctuations and we study their interaction with the anisotropic background, searching for imprints on the temporal evolution of the perturbations' amplitude.

Confining ourselves in the limit of zero resistivity (infinite conductivity), we solve analytically the system of partial differential equations which governs the evolution of the magnetized cosmological perturbations. Our results verify that, fast-magnetosonic modes are excited within the ideal plasma. But, what's most important, is that, at late times, the magnetic induction contrast $(\delta B/B)$ grows, following a power-law temporal dependence, thus leading to the enhancement of the ambient magnetic field (dynamo effect), with the linear approximation being valid at high accuracy. This effect is particularly favored by condensations that can be formed within the anisotropic fluid, due to a gravitational instability, since, as we recover, the magnetic induction contrast amplifies in tune with the corresponding energy-density counterpart.

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